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Conservation of mechanical energy and circulation in the theory of inviscid fluid sheets

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Abstract. In the theory of thin fluid sheets, governing equations are derived with specific reference to an assumed simple kinematic structure of the flow. There is a separate set of governing equations associated with each degree of complexity of the kinematic structure, forming a hierarchy of models (Green and Naghdi [3] and Shields and Webster [8]). If one is interested in the velocity profile across the sheet, the kinematic structure can be used again to interpret the variables in the governing equations as an approximate flow. This paper is concerned with the properties of this approximate flow.

Two important consequences of the field equations (Euler's equations) in the classical, three-dimensional theory of ideal fluids are: conservation of mechanical energy, and conservation of circulation (Kelvin's theorem). The research reported herein provides a proof that mechanical energy is exactly conserved for the approximate flow in each level in this hierarchy. Two types of circulation are considered in the approximate flow: "in-sheet" circulation which is computed about circuits lying a fixed fractional distance between the top and bottom surfaces of the sheet, and "cross-sheet" circulation which is computed about circuits lying in a vertical cylindrical surface. It was found that K moments of the in-sheet circulation and $K - 1$ weighted moments of the cross-sheet circulation are conserved in the Kth level approximate flow.

1. Introduction

The theory of fluid sheets has been developed in recent years for describing the behavior of a thin layer of an inviscid and incompressible fluid exposed to a variety of different flow situations. The approximate governing equations for such fluid sheets have been derived either using the Kantorovich variational method (Levich and Krylov [4]; Shields [7]), using Hamilton's Principle (Miles and Salmon [5]) or using a direct formulation (Green and Naghdi [1-3]). In any case, there is no single set of governing equations. Rather, depending on the assumed kinematical structure of the flow, a hierarchy of sets of governing equations is produced. Miles and Salmon derive only the first level in this hierarchy (which Green and Naghdi refer to as "restricted theory"). They exploit the assumed kinematic structure for this level in which fluid particles originally within a vertical column move horizontally as a unit and retain their columnar character. For higher-level theories, the corresponding kinematic structure does not lead to columnar motion, and it is not clear that the M-S approach can be extended to a situation where columns are not preserved.

Before we discuss conservation of either energy or circulation, it is useful to give a brief overview of the theory. The presentation here reflects the L-K approach described by Shields and Webster [8]. The theory is developed with the aid of an approximation to the kinematics of the real flow field, in which the form of the fluid-velocity variation across the sheet is specified. Referring to Figure 1, we take a coordinate system with x^1 and x^2 horizontal and x^3 vertically upwards. The fluid sheet is assumed to be bounded by material surfaces $x^3 = \alpha(x^1, x^2, t)$

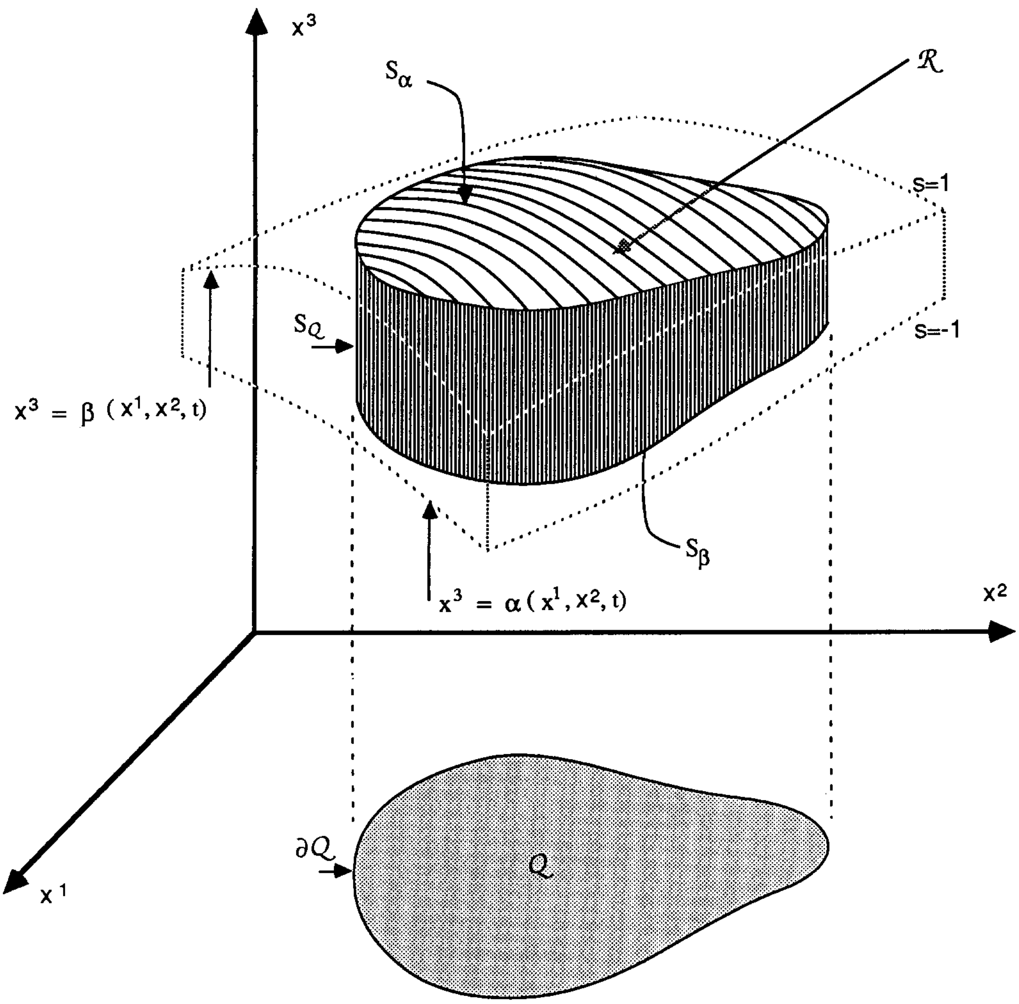

Fig. 1. Definition of the fluid sheet.

and $x^3 = \beta(x^1, x^2, t)$. The local thickness, $\eta(x^1, x^2, t)$, and the location of the midsurface of the sheet, $\zeta(x^1, x^2, t)$, are given by $\beta - \alpha$ and $(\alpha + \beta)/2$, respectively. For the Kth level of the hierarchy, the fluid velocity $\mathbf{v}^*$ is assumed to have the following polynomial form†

$$\mathbf{v}^*(x^1, x^2, x^3, t) = \sum_{n=0}^{K} \mathbf{w}_n(x^1, x^2, t) \cdot s^n, \tag{1.1}$$

where

$$\mathbf{w}_K(x^1, x^2, t) = w_K^3(x^1, x^2, t)\mathbf{e}^3,$$

† We use standard vector and tensor notation. Components of vectors are indicated with superscripts, whereas the vector itself is indicated by a **bold** character. Derivatives are indicated by subscripts preceded by a comma. Lower case Latin indices take on values 1, 2 & 3, lower case Greek indices take on values of 1 & 2, and repeated indices indicate summation.

$s = 2(x^3 - \zeta)/\eta$ is a nondimensional coordinate which varies from -1 on the bottom of the sheet to $+1$ on the top of the sheet, and where each $\mathbf{w}_n$ has the dimension of velocity. In the terminology of Green and Naghdi, the $\mathbf{w}_n$ are referred to as "directors". The second equation in (1.1) states that the director $\mathbf{w}_K$ is restricted to the x^3 direction. A three-dimensional pressure distribution, $p^*(x^1, x^2, x^3, t)$, is associated with this kinematic approximation. We define the pressures on the top and bottom of the sheet as $\hat{p} = p^*|_{x^3=\beta(x^1,x^2,t)}$ and $\bar{p} = p^*|_{x^3=\alpha(x^1,x^2,t)}$, respectively.

The general dynamical equations for the fluid sheet, derived with reference to (1.1), consist of three ingredients: (a) 2 kinematic boundary conditions, one each for the top and bottom surfaces, (b) K continuity equations, expressing the incompressibility of the fluid, and (c) $3K + 1$ approximate momentum equations (K vector equations and one scalar equation). These equations are *not* exact equations of motion for the flow given in (1.1). Although the kinematic boundary conditions and the continuity conditions are exact conditions for the flow given in (1.1), the momentum equations state only that the integral of the momentum and of K moments in s of the momentum are preserved across the sheet. There are $2K + 1$ independent moments in s of the momentum for the flow in (1.1) and these are easily written down. To do so, however, would yield more equations than unknowns. This result is to be expected, of course, since we should not suppose that flows of even very thin fluid sheets can be expressed exactly by the simple polynomial form given in (1.1). As an aside, we note that there is more than one way to achieve the same number of equations as unknowns when using the Kantorovich procedure. For instance, it would have been possible to obtain alternate sets of governing equations which are mathematically well-posed by using more approximate momentum equations and fewer approximate conservation-of-mass relations. Levich and Krylov remove this potential ambiguity by specifying that conservation of mass be satisfied exactly.

The general dynamical equations are two-dimensional equations involving variables which are functions of x^1, x^2 and t only. Typically, the sought-after quantities for most problems are displayed directly in terms of those variables which deal with the bounding surfaces of the sheet. For instance, in water-wave problems the variable η yields the wave profile and wave speed and the variable $\bar{p}$ gives the pressure at the bottom. In some problems an approximation to the velocity profile within the sheet is also desired and (1.1) is used again to provide an interpretation of the directors, $\mathbf{w}_n$, appearing in the mathematical model. This step could perhaps be regarded as an additional assumption, as it is separate from those used in the derivation of the model itself. Since the exact momentum equations are not satisfied, this approximate flow is not necessarily a realizable ideal flow. (Under any circumstances, the theory does not yield a pressure distribution across the sheet, but rather gives only the integral of the pressure and $K - 1$ moments in s of the pressure at any given x^1, x^2.)

The remainder of this paper is aimed at examining the approximate flow resulting from interpreting the variables in the governing equations using (1.1) and, in particular, comparing several features of this flow with known features of ideal fluid flows. In classical mechanics of an ideal fluid two important consequences of the momentum equations (Euler's equations) and the conservation of mass are the conservation of mechanical energy and the conservation of circulation (Kelvin's theorem). We assume that the fluid is isothermal so that thermodynamics is not involved here. It is of interest to ask whether these two quantities are conserved in the approximate flow. Since only a subset of the three-dimensional momentum equations is satisfied in the case of thin fluid sheets, conservation of energy and vorticity in the approximate flow is, in principle, an open question. Conservation of mechanical energy is important if we wish to compute, say, quantities such as wave drag; conservation of circulation is important if we wish to compute, say, water waves or geophysical flows.

It has been shown (Green and Naghdi [1]) that the average mechanical energy across the sheet in G–N theory is always preserved, even when the kinematic approximation to the flow is not a power series, such as given in (1.1). In the M–S approach, a Lagrangian energy density from the kinematic approximation is used to derive equations of motion and, thus, conservation of mechanical energy is assured.

The question of mechanical-energy conservation is not addressed in the L–K approach. Shields [7] has shown that for ideal fluids and for the same level of the hierarchy, the governing equations resulting from the G–N approach and those from the L–K approach are algebraically transformable one to another, and are thus identical in substance. Therefore, since the G–N equations do conserve mechanical energy, the L–K equations must also. The demonstration of the transformation between the two sets of governing equations is a tedious exercise; the direct demonstration of conservation of mechanical energy is simpler and is given in Section 3. Since this latter derivation also gives some insight into the physical nature of the approximation, we present it here.

For two-dimensional ideal-fluid flow, vorticity is transported with the fluid particles, as can be easily seen by taking the curl of the momentum equations. In three-dimensional flow the same operation leads to the so-called "vortex-stretching" terms. The vorticity measured at a particle does not, in general, remain constant unless the vorticity is zero everywhere at one instant in time, and thus remains zero everywhere for all time. Miles and Salmon [5] consider the Ertel potential vorticity for the first-level theory, the only level they treat. They derive a conservation law for the sum of the vertical and a nonvertical component of potential vorticity (ζ/h and ζ^*/h in their notation, respectively). ζ/h corresponds to the vertical vorticity that one computes in ordinary shallow-water theory (where the flow is nearly horizontal). The decomposition into these two components, however, may not be useful for flows with a considerable vertical orientation, such as a waterfall. In the approach presented below, we use the sheet itself to provide an orientation for the treatment of circulation, and this results in a considerable simplification.

In all but the first-level theory, all components of the fluid-velocity vector vary across the sheet. The approximate flow is thus a complex three-dimensional flow, and it is more in keeping with the results of the classical theory of ideal fluids to examine conservation of circulation rather than conservation of vorticity. In Section 4 below we investigate the conservation of circulation for contours which lie within the sheet initially at a fixed fractional distance between the upper and lower surfaces, and that of contours which lie in a vertical cylinder passing through the sheet. We will refer to the former as "in-sheet" circulation and the latter by "cross-sheet" circulation.

2. The governing equations

The governing equations for the Kth level in the thin-sheet hierarchy are presented in Shields and Webster [8]. If the Kth vector in (1.1) is restricted ($w_K^y \equiv 0$), then the equations of motion are:

Kinematic boundary conditions†:

$$2\sum_{n=1,3}^{K} w_n^3 = \eta_{,t} + \eta_{,\gamma}\sum_{n=0,2}^{K-1} w_n^\gamma + 2\zeta_{,\gamma}\sum_{n=1,3}^{K-1} w_n^\gamma,$$

and (2.1)

$$2\sum_{n=0,2}^{K} w_n^3 = 2\zeta_{,t} + \eta_{,\gamma}\sum_{n=1,3}^{K-1} w_n^\gamma + 2\zeta_{,\gamma}\sum_{n=0,2}^{K-1} w_n^\gamma;$$

Continuity:

$$\zeta_{,\gamma} w_n^\gamma + \tfrac{1}{2}\eta_{,\gamma} w_{n-1}^\gamma - w_n^3 = \frac{1}{2n}(\eta w_{n-1}^\gamma)_{,\gamma}, \quad n = 1, 2, \ldots, K; \tag{2.2}$$

Momentum:

$$\begin{aligned}(\eta/2)\int_{-1}^{1} s^n \mathscr{D}\mathbf{v}^*\, \mathrm{d}s &= \sum_{m=0}^{K}\Big\{\eta\theta_{m+n}[\mathbf{w}_{m,t} + w_0^\gamma \mathbf{w}_{m,\gamma}] \\ &\quad + \sum_{r=1}^{K-1}\theta_{m+n+r}\,[(\eta w_r^\gamma)\mathbf{w}_{m,\gamma} + m\mu^r_{m+n}(\eta w\gamma_r)_{,\gamma}\mathbf{w}_m]\Big\} \\ &= -(1/\varrho)\,\{[p_{n,\gamma} + np_n(\eta_{,\gamma}/\eta) + 2np_{n-1}(\zeta_{,\gamma}/\eta) - \hat{p}\beta_{,\gamma} \\ &\quad + (-1)^n\bar{p}\alpha_{,\gamma}]\mathbf{e}^\gamma + [\hat{p} - (-1)^n\bar{p} - 2np_{n-1}/\eta + \theta_n\varrho g\eta]\mathbf{e}^3\}, \\ &\quad n = 0, 1, \ldots, K-1,\end{aligned} \tag{2.3}$$

and for $n = k$

$$\begin{aligned}(\eta/2)\int_{-1}^{1} s^K \mathscr{D}\mathbf{v}^{3*}\, \mathrm{d}s &= \sum_{m=0}^{K}\Big\{\eta\theta_{m+K}[\mathbf{w}^3_{m,t} + w_0^\gamma \mathbf{w}^3_{m,\gamma}] \\ &\quad + \sum_{r=1}^{K-1}\theta_{m+K+r}\,[(\eta w_r^\gamma)\mathbf{w}^3_{m,\gamma} + m\mu^r_{m+K}(\eta w_r^\gamma)_{,\gamma}\mathbf{w}^3_n]\Big\} \\ &= -(1/\varrho)[\hat{p} - (-1)^K\bar{p} - 2Kp_{K-1}/\eta + \theta_K\varrho g\eta]\mathbf{e}^3,\end{aligned} \tag{2.3a}$$

† The notation $\Sigma^K_{n=1,3}$ indicates a summation over odd values of $n \leqslant K$, beginning with $n = 1$, and $\Sigma^K_{n=0,2}$ indicates a summation over even values of $n \leqslant K$, beginning with $n = 0$.

where

$$\theta_s = \begin{cases} 1/(s+1), & s \text{ even} \\ 0, & s \text{ odd} \end{cases} \qquad \mu_s^r = \begin{cases} 1/s, & r \text{ odd}, \\ r/\{(r+1)\cdot(s+1)\}, & r \text{ even}, \end{cases}$$

$$p_n = (\eta/2)\int_{-1}^{1} s^n p^* \, \mathrm{d}s,$$

and where $\mathscr{D}\mathbf{a} = \mathbf{a}_{,t} + v^{*i}\mathbf{a}_{,i}$ is the material derivative of an arbitrary vector $\mathbf{a}$.

3. Conservation of mechanical energy

Consider a region $\mathscr{R}$ formed by the intersection of the fluid sheet with a vertical cylinder generated by lines parallel to the x^3 axis passing through the curve $\partial\mathscr{Q}(x^1, x^2, t)$ lying in the (x^1, x^2) plane (see Fig. 1). $\partial\mathscr{R}$, the boundary of $\mathscr{R}$, is composed of three surfaces: $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$, the bottom and top surfaces of the fluid sheet, and $\mathscr{S}_{\mathscr{Q}}$, the vertical surface formed by their intersection with the cylinder. The total energy of the flow contained within $\mathscr{R}$ is the sum of the kinetic and potential energies, and is given by

$$\mathscr{E}^*(t) = \iiint_{\mathscr{R}} [\tfrac{1}{2}|\mathbf{v}^*|^2 + gx^3]\varrho \, \mathrm{d}\mathscr{R}. \tag{3.1}$$

From approximate flow given in (1.1), the square of the magnitude of the three-dimensional velocity given in (3.1) is

$$\tfrac{1}{2}|\mathbf{v}^*|^2 = \sum_{n=0}^{K}\sum_{m=0}^{K} \tfrac{1}{2}(\mathbf{w}_n \cdot \mathbf{w}_m)s^{n+m}$$

and from the definition of s, we can write $x^3 = \zeta + s\eta/2$, and the differential volume $\mathrm{d}\mathscr{R} = (\eta/2)\,\mathrm{d}s\,\mathrm{d}x^1\,\mathrm{d}x^2$. Inserting these expressions into (3.1) and exploiting symmetry, we get

$$\mathscr{E}^*(t) = \iint_{\mathscr{Q}} \mathrm{d}x^1\,\mathrm{d}x^2 \int_{-1}^{+1} \mathrm{d}s\,\tfrac{1}{2}\varrho\eta\left\{\sum_{n=0}^{K}\sum_{m=n}^{K} \tfrac{1}{2}(2-\delta_{nm})(\mathbf{w}_n\cdot\mathbf{w}_m)s^{n+m} + g[\zeta + s\eta/2]\right\}. \tag{3.2}$$

The integration over s can be performed explicitly yielding

$$\mathscr{E}^*(t) = \iint_{\mathscr{Q}} \mathrm{d}x^1\,\mathrm{d}x^2 \varrho\eta\mathscr{T}, \tag{3.3}$$

where

$$\mathscr{T} = \sum_{n=0}^{K}\sum_{m=n}^{K} \tfrac{1}{2}(2-\delta_{nm})\theta_{m+n}(\mathbf{w}_n\cdot\mathbf{w}_m) + g\zeta.$$

Consider now the time of change of the total energy enclosed with $\mathscr{R}$. From Leibnitz' theorem, the derivative of (3.3) with respect to time is

$$\mathscr{E}^*(t)_{,t} = \iint_{\mathscr{G}} \mathrm{d}x^1\, \mathrm{d}x^2 \varrho(\eta\mathscr{T})_{,t} + \oint_{\partial\mathscr{Q}} \mathrm{d}\sigma\varrho\eta\mathscr{T}V_n, \tag{3.4}$$

where $\mathrm{d}\sigma$ is the differential arc length and V_n is the outward normal velocity of a point on $\partial\mathscr{R}$ which is the same as the outward normal velocity on the curve $\partial\mathscr{Q}$.

In order to proceed, we need to find an expression for $(\eta\mathscr{T})_{,t}$ and to do this we must appeal to the mathematical model for the flow (2.1–2.3). As a preliminary, we need two results easily derived from the kinematic boundary conditions and continuity (2.1–2.2). First, Shields [7] showed that these two sets of equations can be combined to yield a compact expression for $\eta_{,t}$. A similar expression for $\zeta_{,t}$ can also be found. We record these expressions here:

$$\begin{aligned} \eta_{,t} &= -\sum_{n=0,2}^{K} [1/(n+1)](\eta w_n^\gamma)_{,\gamma}, \\ \zeta_{,t} &= w_0^3 - \zeta_{,\gamma} w_0^\gamma - \sum_{n=1,3}^{K} \frac{1}{2(n+1)} (\eta w_n^\gamma)_{,\gamma}. \end{aligned} \tag{3.5}$$

Second, for any function $\mathscr{H}(x^1, x^2, t)$ we have

$$\eta(\mathscr{H}_{,t} + w_0^\gamma \mathscr{H}_{,\gamma}) = \eta\mathscr{H}_{,t} + \eta(w_0^\gamma\mathscr{H})_{,\gamma} - \eta\mathscr{H}w_{0,\gamma}^\gamma. \tag{3.6}$$

Inserting the first equation of (3.5) into (3.6), we obtain

$$\eta(\mathscr{H}_{,t} + w_0^\gamma \mathscr{H}_{,\gamma}) = \eta\mathscr{H}_{,t} + \eta(w_0^\gamma\mathscr{H})_{,\gamma} + \mathscr{H}\eta_{,t} + \eta_{,\gamma}(w_0^\gamma\mathscr{H}) + \mathscr{H}\sum_{n=2,4}^{K-1} \theta_n(\eta w_n^\gamma)_{,\gamma}$$

or

$$\eta(\mathscr{H}_{,t} + w_0^\gamma \mathscr{H}_{,\gamma}) = (\eta\mathscr{H})_{,t} + (\eta w_0^\gamma\mathscr{H})_{,\gamma} + \mathscr{H}\sum_{n=2,4}^{K-1} \theta_n(\eta w_n^\gamma)_{,\gamma}. \tag{3.7}$$

To obtain a general expression for $(\eta\mathscr{T})_{,t}$, we begin by taking the dot product of the nth momentum equations (2.3) and (2.3a) with $\mathbf{w}_n$ and summing. Since in (1.1) the components w_K^γ are identically zero, this summation can be written compactly as

$$\begin{aligned} &\sum_{n=0}^{K}\sum_{m=0}^{K} \Big\{\tfrac{1}{2}\eta\theta_{m+n}[(\mathbf{w}_n\cdot\mathbf{w}_m)_{,t} + w_0^\gamma(\mathbf{w}_n\cdot\mathbf{w}_m)_{,\gamma}] \\ &\qquad + \sum_{r=1}^{K-1} \theta_{m+n+r}\, [(\eta w_r^\gamma)\mathbf{w}_n\cdot\mathbf{w}_{m,\gamma} + m\mu_{m+n}^r(\eta w_r^\gamma)_{,\gamma}\mathbf{w}_n\cdot\mathbf{w}_m]\Big\} \\ &= -(1/\varrho)\sum_{n=0}^{K} \{[p_{n,\gamma} + np_n(\eta_{,\gamma}/\eta) + 2np_{n-1}(\zeta_{,\gamma}/\eta) - \hat{p}\beta_{,\gamma} + (-1)^n\bar{p}\alpha_{,\gamma}]w_n^\gamma \\ &\qquad + [\hat{p} - (-1)^n\bar{p} - 2np_{n-1}/\eta + \theta_n\varrho g\eta]w_n^3\}. \end{aligned} \tag{3.8}$$

It is convenient to consider the right- and left-hand sides of (3.8) separately. Consider first the left-hand side. Using (3.7), exploiting symmetry to make the summation over m range from n to K, and expanding the summation over r into odd and even components, we obtain

$$\sum_{n=0}^{K}\sum_{m=n}^{K}\Bigg\{\tfrac{1}{2}(2-\delta_{nm})\theta_{m+n}[(\eta(\mathbf{w}_n\cdot\mathbf{w}_m)_{,t}+(\eta w_0^{\gamma}\mathbf{w}_n\cdot\mathbf{w}_m)_{,\gamma}]$$
$$+\sum_{r=2,4}^{K-1}\tfrac{1}{2}(2-\delta_{nm})[\theta_{m+n}\theta_r(\eta w_r^{\gamma})_{,\gamma}(\mathbf{w}_n\cdot\mathbf{w}_m)+\theta_{m+n+r}(\eta w_r^{\gamma})(\mathbf{w}_m\cdot\mathbf{w}_m)_{,\gamma}$$
$$+\theta_{m-n+r}(m+n)\mu_{m+n}^{r}(\eta w_r^{\gamma})_{,\gamma}(\mathbf{w}_n\cdot\mathbf{w}_m)]$$
$$+\sum_{r=1,3}^{K-1}\theta_{m+n+r}\,[(\eta w_r^{\gamma})(\mathbf{w}_n\cdot\mathbf{w}_m)_{,\gamma}+(\eta w_r^{\gamma})_{,\gamma}(\mathbf{w}_n\cdot\mathbf{w}_m)]\Bigg\},\tag{3.9}$$

where δ_{nm} is the Kronecker delta, and in the last summation use has been made of the identity $(m+n)\mu_{m+n}^{r}=1$ when r is odd. When r is even, the first and third terms in the second summation can be combined using the identity

$$\theta_{m+n}\theta_r = \theta_{m+n+r}-(m+n)\theta_{m+n+r}\mu_{m+n}^{r}.$$

With this identity, the summand in the second summation of (3.9) is seen to be a divergence. Finally, recombining the odd and even summations with the last term of the first line of (3.9), the left-hand side can be written compactly as

$$\sum_{n=0}^{K}\sum_{m=n}^{K}\tfrac{1}{2}(2-\delta_{nm})\Bigg\{\theta_{m+n}(\eta\mathbf{w}_n\cdot\mathbf{w}_m)_{,t}+\sum_{r=0}^{K-1}\theta_{m+n+r}(\eta w_r^{\gamma}\mathbf{w}_n\cdot\mathbf{w}_m)_{,\gamma}\Bigg\}.\tag{3.10}$$

We turn our attention now to the right-hand side of (3.8). Collecting terms we obtain

$$+(1/\varrho)\bar{p}\sum_{n=0}^{K}(\beta_{,\gamma}w_n^{\gamma}-w_n^{3})-(1/\varrho)\bar{p}\sum_{n=0}^{K}(-1)^n(\alpha_{,\gamma}w_n^{\gamma}-w_n^{3})-g\eta\sum_{n=0,2}^{K}1/(n+1)w_n^{3}$$
$$-(1/\varrho)\sum_{n=1}^{K}\{p_{n-1,\gamma}w_{n-1}^{\gamma}+(n-1)p_{n-1}w_{n-1}^{\gamma}(\eta_{,\gamma}/\eta)+2nw_n^{\gamma}p_{n-1}(\zeta_{,\gamma}/\eta)-2nw_n^{3}p_{n-1}/\eta\}.\tag{3.11}$$

As a result of the kinematic boundary conditions (2.1) and the definitions of α and β, the first two summations in (3.11) can be written

$$\sum_{n=0}^{K}(\beta_{,\gamma}w_n^{\gamma}-w_n^{3}) = -\beta_{,t},\quad\text{and}\quad\sum_{n=0}^{K}(-1)^n(\alpha_{,\gamma}w_n^{\gamma}-w_n^{3}) = -\alpha_{,t}.\tag{3.12}$$

The third summation in (3.11) is more complicated and requires both the kinematic boundary condition and the continuity conditions. Multiplying the nth continuity equation by $[n/(n+1)]$ and summing over even values of n from 2 to K, we obtain

$$\sum_{n=2,4}^{K} \frac{n}{n+1} w_n^3 = \eta_{,y} \sum_{n=2,4}^{K} \frac{n}{2(n+1)} w_{n-1}^{y} + \zeta_{,y} \sum_{n=2,4}^{K} \frac{n}{n+1} w_n^{y}$$
$$- \sum_{n=2,4}^{K} \frac{1}{2(n+1)} (\eta w_{n-1}^{y})_{,y}. \tag{3.13}$$

Replacing n with $n + 1$ in the first and third summation of (3.13) makes these summations of w_n^y over odd values of n from 1 to $K - 1$. Dividing the second kinematic boundary condition in (2.1) by 2, subtracting (3.13) from it, and multiplying the result by η yields

$$\eta \sum_{n=0,2}^{k} \frac{1}{n+1} w_n^3 = \eta(\zeta_{,t} + \zeta_{,y} w_0^y) + \eta\zeta_{,y} \sum_{n=2,4}^{K} \frac{1}{(n+1)} w_n^y + \sum_{n=1,3}^{K-1} \frac{1}{2(n+2)} (\eta^2 w_n^y)_{,y}. \tag{3.14}$$

The first two terms on the right-hand side of (3.14) can be replaced using (3.7). Combining the resulting terms, we obtain

$$\eta \sum_{n=0,2}^{K} \frac{1}{(n+1)} w_n^3 = (\zeta\eta)_{,t} + \sum_{n=0,2}^{K} \frac{1}{n+1} (\eta\zeta w_n^y)_{,y} + \sum_{n=1,3}^{K-1} \frac{1}{2(n+2} (\eta^2 w_n^y)_{,y}. \tag{3.15}$$

Finally, we consider the last term in (3.11). Solving the continuity equation (3) for w_n^3, and eliminating this variable from the last term in (3.11), we obtain

$$-(1/\varrho) \sum_{n=1}^{K} \{p_{n-1,y} w\gamma_{n-1} + p_{n-1}(\eta w_{n-1}^y)_{,y}/\eta - p_{n-1} w\gamma_{n-1}\gamma_{,y}/\eta)\} = -(1/\varrho) \sum_{n=1}^{K} (\varrho_{n-1} w_{n-1}^y)_{,y}. \tag{3.16}$$

Assembling the results given in Equations (3.10–12), (3.15) and (3.16), we get

$$\sum_{n=0}^{K} \sum_{m=n}^{K} \tfrac{1}{2}(2 - \delta_{nm})\theta_{m+n}(\eta \mathbf{w}_n \cdot \mathbf{w}_m)_{,t} + g(\zeta\eta)_{,t}$$
$$= -\sum_{n=0}^{K} \sum_{m=n}^{K} \sum_{r=0}^{K-1} \tfrac{1}{2}(2 - \delta_{nm})\theta_{m+n+r}(\eta w_r^y \mathbf{w}_n \cdot \mathbf{w}_m)_{,y}$$
$$- (1/\varrho) \sum_{n=1}^{K} (p_{n-1} w_{n-1}^y)_{,y} - g \sum_{n=0,2}^{K} \frac{1}{n+1} (\eta\zeta w_n^y)_{,y}$$
$$- g \sum_{n=1,3}^{K-1} \frac{1}{2(n+2)} (\eta^2 w_n^y)_{,y} + (\bar{p}\alpha_{,t} - \hat{p}\beta_{,t})/\varrho. \tag{3.17}$$

The left-hand side of (3.17) is precisely $(\eta\mathcal{T})_{,t}$ and we can replace it by the right-hand side, which involves no time derivatives. Since each summation on the right-hand side of (3.17) is a divergence, we can use the divergence theorem to express (3.4) as

$$\begin{aligned}
\mathscr{E}^*(t)_{,t} \;=\; & -\left\{\iint_{\mathscr{S}_\alpha} \bar{p}V_n \,\mathrm{d}\mathscr{S}_\alpha + \iint_{\mathscr{S}_\beta} \hat{p}V_n \,\mathrm{d}\mathscr{S}_\beta\right\} \\
& + \oint_{\partial\mathscr{Q}} \mathrm{d}\sigma\, \varrho\eta V_n \left\{\sum_{n=0}^{K}\sum_{m=n}^{K} \tfrac{1}{2}(2-\delta_{nm})\theta_{m+n}(\mathbf{w}_n\cdot\mathbf{w}_m) + g\zeta\right\} \\
& + \oint_{\partial\mathscr{Q}} \mathrm{d}\sigma \left[\sum_{r=0}^{K-1} p_r v_r^n - \varrho g\left\{\sum_{r=0,2}^{K} \frac{1}{n+1}\eta\zeta v_r^n + \sum_{r=1,3}^{K-1} \frac{1}{2(n+1)}\eta^2 v_r^n\right\}\right. \\
& \left. + \varrho\eta \sum_{n=0}^{K}\sum_{m=n}^{K} \tfrac{1}{2}(2-\delta_{mn}) \sum_{r=0}^{K-1} \theta_{m+n+r} v_r^n \mathbf{w}_n\cdot\mathbf{w}_m\right]
\end{aligned} \tag{3.18}$$

where v_r^n is the "normal" velocity component of the $\mathbf{w}_r$ vector defined by $\mathbf{w}_r \cdot \mathbf{n}$, and $\mathbf{n}$ is the outward normal vector to $\partial\mathscr{Q}$.

Equation (3.18) is easily interpreted. It states that the time rate of change of the total energy in the region $\mathscr{R}$ (the left-hand side) is equal to: the work done by the external pressures on the surfaces $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$ (the first term on the right-hand side), the total energy captured (or lost) as a result of the motion of the non-material surface $\mathscr{S}_\mathscr{Q}$ (the second term), the work done by internal pressures on $\mathscr{S}_\mathscr{Q}$ (the third term), the convection of potential energy into $\mathscr{R}$ through $\mathscr{S}_\mathscr{Q}$ (the fourth term), and, finally, the convection of kinetic energy through $\mathscr{S}_\mathscr{Q}$ (the fifth term).

As a consequence, we see that the approximate equations for the fluid sheet (2.1–2.3) preserve energy in the approximate flow in the usual sense. In particular, if the surfaces $\mathscr{S}_\alpha$ and $\mathscr{S}_\beta$ are either fixed surfaces or are free surfaces with zero pressure acting on them, then the only change in energy in the region $\mathscr{R}$ occurs through "leakage" through $\mathscr{S}_\mathscr{Q}$. If, in addition $\mathscr{S}_\mathscr{Q}$ is a surface of no fluid motion then $\mathcal{T}$ remains constant throughout the flow region.

4. Conservation of circulation

4.1. In-sheet circulation

Consider a closed contour, $\partial\mathscr{Q}$, lying in the (x^1, x^2) plane as shown in Figure 2. We define a coordinate, λ, along this contour with $\lambda = 0$ and $\lambda = \ell$ corresponding to the same point. With ∂Q as a generating curve we can define a separate contour, C_s, in the fluid sheet for a given value of the nondimensional vertical coordinate, s. C_s can be expressed parametrically by $x^1(\lambda, t)$, $x^2(\lambda, t)$, and $x^3(\lambda, s, t) \equiv \zeta(\lambda, t) + s\eta(\lambda, t)/2$, with $x^i(0, t) = x^i(\ell, t)$. The circulation $\Gamma(s, t)$ about this contour is defined as

$$\Gamma(s, t) \;=\; \int_0^{\ell} \mathbf{v}^* \cdot \mathbf{t}\, \mathrm{d}\sigma \;=\; \int_0^{\ell} \mathbf{v}^* \cdot \mathbf{x}_{,\lambda}\, \mathrm{d}\lambda \tag{4.1}$$

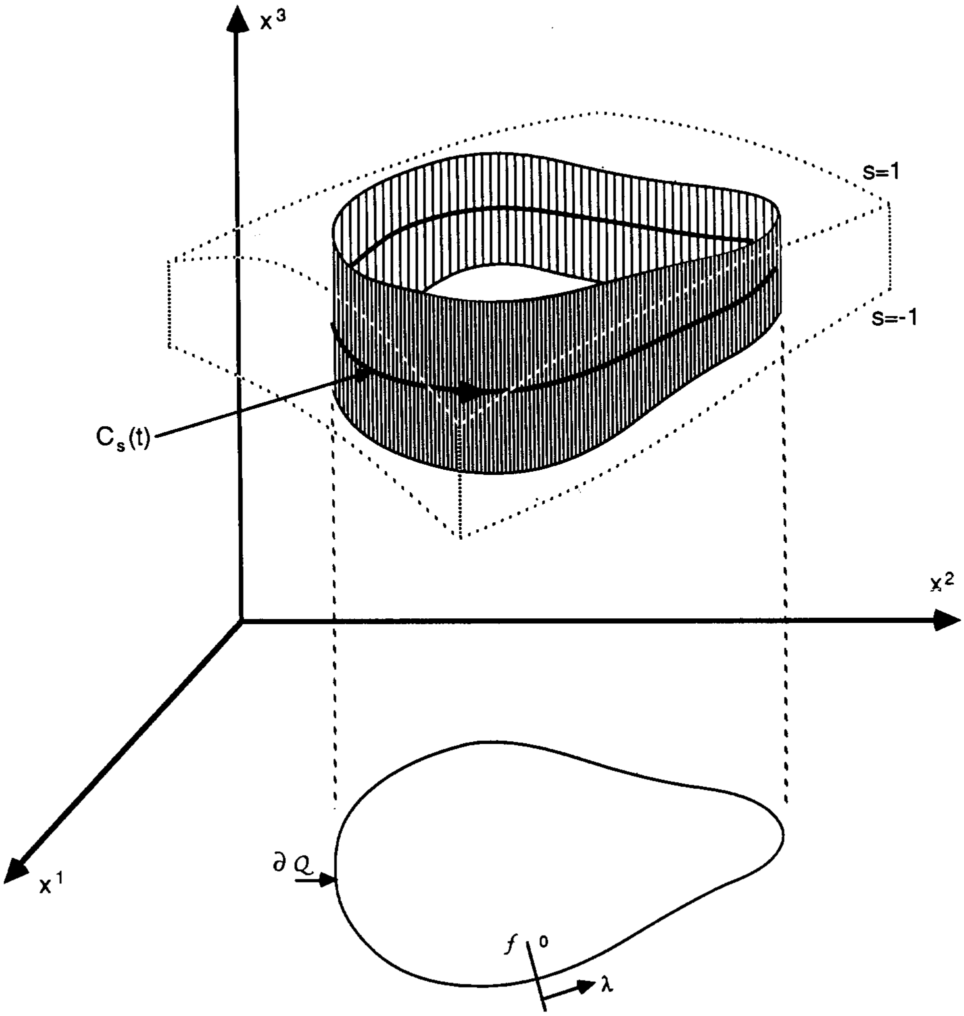

Fig. 2. Definition of the contour $C_s(t)$ used in the computation of in-sheet circulation.

where $\mathbf{v}^*(\lambda, s, t)$ is the fluid velocity, $\mathbf{t}(\lambda, s, t)$ is the tangent vector, and $\mathbf{t}\ \mathrm{d}\sigma = \mathbf{x}_{,\lambda}\ \mathrm{d}\lambda$ is the differential arc length, all evaluated on C_s. We compute now $\Gamma_{,t}$ following the particles on the contour C_s. That is, the coordinate λ is used only to mark these particles at a given instant of time and is therefore not a function of time itself. The derivative of (4.1) with respect to time yields

$$\Gamma_{,t} = \int_0^{f} \{\mathscr{D}\mathbf{v}^* \cdot \mathbf{x}_{,\lambda} + \mathbf{v}^* \cdot \mathbf{v}^*_{,\lambda}\}\ \mathrm{d}\lambda \tag{4.2}$$

where use has been made of the relation $\mathbf{x}_{,t} = \mathbf{v}^*$ when following particles. Since $\mathbf{v}^* \cdot \mathbf{v}^*_{,\lambda} = \frac{1}{2}(\mathbf{v}^*)^2_{,\lambda}$, and $\mathbf{v}^*(0, s, t) = \mathbf{v}^*(f, s, t)$, (4.2) becomes simply

$$\Gamma_{,t} = \int_0^{f} \mathscr{D}\mathbf{v}^* \cdot \mathbf{x}_{,\lambda}\ \mathrm{d}\lambda. \tag{4.3}$$

Each value of the parameter s, $-1 \leqslant s \leqslant 1$, corresponds to a single curve on the strip. We now compute $\tilde{\Gamma}^n_{,t}$, the nth moment in s of $\Gamma_{,t}$, defined by

$$\tilde{\Gamma}^n_{,t} = \frac{1}{2}\int_{-1}^{1} s^n \, ds \int_0^{\ell} \mathscr{D}\mathbf{v}^* \cdot \mathbf{x}_{,\lambda} \, d\lambda, \quad n = 0, 1, \ldots, K-1. \tag{4.4}$$

$x^1(\lambda, t)$, $x^2(\lambda, t)$, $\zeta(\lambda, t)$ and $\eta(\lambda, t)$ are all independent of s. Interchanging orders of integration in (4.4) and using the relation $x^3(\lambda, s, t) \equiv \zeta(\lambda, t) + s\eta(\lambda, t)/2$ yields

$$\tilde{\Gamma}^n_{,t} = \frac{1}{2}\int_0^{\ell} d\lambda \left\{ x^{\gamma}_{,\lambda} \int_{-1}^{1} s^n \mathscr{D}v^{*\gamma} \, ds + \zeta_{,\lambda} \int_{-1}^{1} s^n \mathscr{D}v^{*3} \, ds + \tfrac{1}{2}\eta_{,\lambda} \int_{-1}^{1} s^{n+1} \mathscr{D}v^{*3} \, ds \right\}. \tag{4.5}$$

The three integrals in (4.5) are exactly $2/\eta$ times the quantities presented as the left-hand sides of equations (2.3). Replacing these integrals with the corresponding right-hand sides yields the following representation for the bracketed terms in (4.5):

$$\begin{aligned} &-[2/(\varrho\eta)]\{[p_{n,\gamma} + np_n(\eta_{,\gamma}/\eta) + 2np_{n-1}(\zeta_{,\gamma}/\eta) - \hat{p}\beta_{,\gamma} + (-1)^n \bar{p}\alpha_{,\gamma}]x^{\gamma}_{,\lambda} \\ &\quad + [\hat{p} - (-1)^n \bar{p} - 2np_{n-1}/\eta + \theta_n \varrho g \eta]\zeta_{,\lambda} \\ &\quad + [\hat{p} - (-1)^{n+1}\bar{p} - 2(n+1)p_n/\eta + \theta_{n+1}\varrho g\eta]\eta_{,\gamma}/2\} \end{aligned} \tag{4.6}$$

Consider now the terms in (4.6). Those involving $\hat{p}$ and $\bar{p}$ are identically zero since

$$\beta_{,\lambda} = \beta_{,\gamma}x^{\gamma}_{,\lambda} = \zeta_{,\lambda} + \eta_{,\lambda/2}, \quad \text{and} \quad \alpha_{,\lambda} = \alpha_{,\gamma}x^{\gamma}_{,\lambda} = \zeta_{,\lambda} - \eta_{,\lambda/2}.$$

The terms involving p_{n-1} cancel out. Collecting the terms involving p_n yields

$$p_{n,\gamma}x^{\gamma}_{,\lambda}/(\varrho\eta) + np_n\eta_{,\gamma}x^{\gamma}_{,\lambda}/(\varrho\eta^2) - (n+1)p_n\eta_{,\gamma}x^{\gamma}_{,\lambda}/(\varrho\eta^2) = (p_n/(\varrho\eta))_{,\lambda}.$$

Inserting these results into (4.6) yields

$$\begin{aligned} \tilde{\Gamma}^n_{,\lambda} &= \int_0^{\ell} d\lambda \, \{-\theta_n g\zeta_{,\lambda} - \theta_{n+1} g\eta_{,\lambda}/2 - (p_n/(\varrho\eta))_{,\lambda}\} \\ &\equiv 0, \quad n = 0, 1, \ldots, K-1, \end{aligned} \tag{4.7}$$

since all of the quantities are perfect differentials and the path is closed.

Equation (4.7) demonstrates that in the Kth level, K moments in s of the in-sheet circulation are exactly preserved. Much of the previous work in the theory of fluid sheets has involved only the first-level theory (referred to as "restricted theory" in the works of Green and Naghdi). In this theory the first moment (the average) circulation remains unchanged. As K becomes large, more moments of the circulation are preserved. Since all of the contours lie in the sheet, the vorticity measured by the above circulation computation is that which is approximately *normal* to the sheet. This approach yields a conservation statement which is somewhat different from that presented by Miles and Salmon [5].

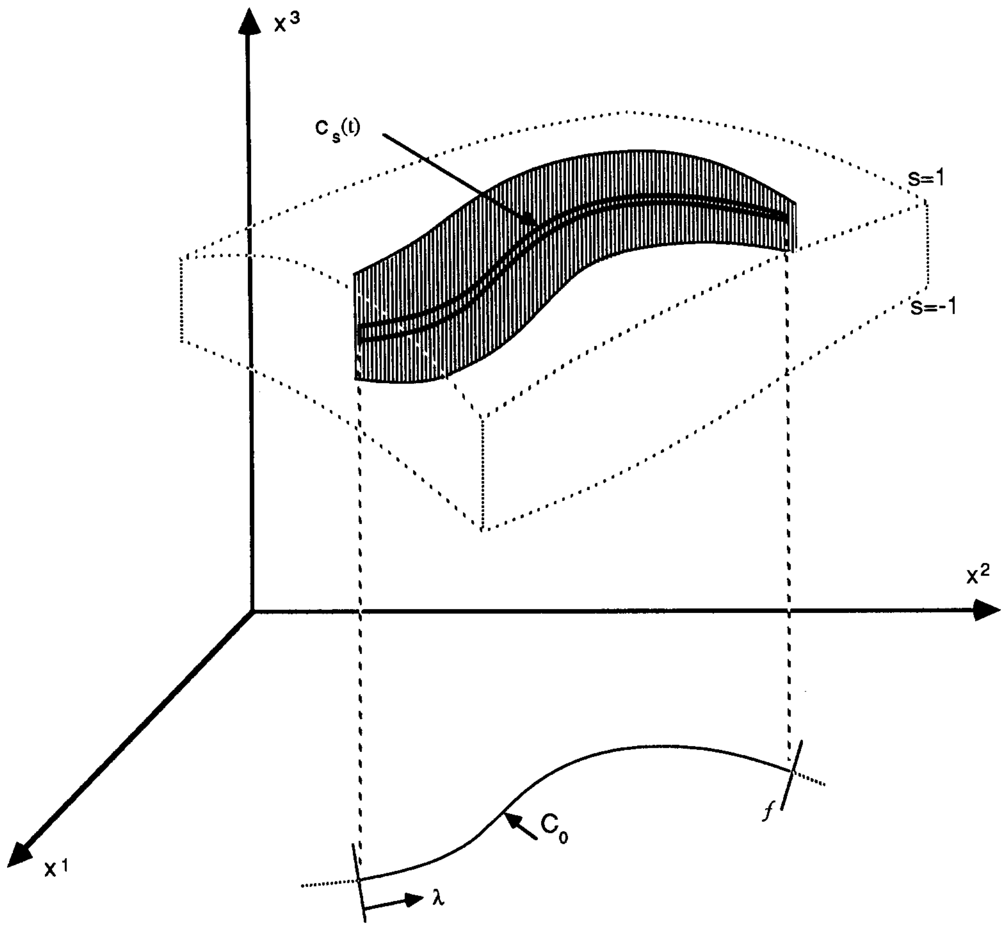

Fig. 3. Definition of the differential contour used in the computation of cross-sheet circulation.

4.2. Cross-sheet circulation

Consider now a curve C_0 lying in the (x^1, x^2) plane and that portion of the surface of a right cylinder generated by this curve which lies within the fluid sheet shown in Figure 3. A coordinate λ along C_0 assumes values from 0 to ℓ. The time rate of change of the circulation, $\hat{\Gamma}_{,t}$, observed by particles originally on a contour C_ℓ spanning the fluid sheet can be expressed by the integral

$$\hat{\Gamma}_{,t} = \frac{1}{2}\int_{-1}^{1} \gamma_{,t}(s)\, \mathrm{d}s,$$

where (4.8)

$$\gamma_{,t}(s) = -\int_0^{\ell} \mathrm{d}\lambda\, \{\mathscr{D}\mathbf{v}^*(\lambda, s, t)\cdot \mathbf{x}_{,\lambda}(\lambda, s, t)\}_{,s} + \{(\eta/2)\mathscr{D}v^{*3}\}|_{\lambda=0}^{\ell},$$

as can be verified by exchanging orders of integration, noting $\mathbf{x}_{,\lambda}\, \mathrm{d}\lambda = (\eta/2)\mathbf{e}^3\, \mathrm{d}s$ on the vertical sides of the contour, and comparing with (4.3). The kernal $\gamma_{,t}(s)\, \mathrm{d}s$ can be identified as the circulation on a differentially narrow contour $c_s(t)$ of width η d$s/2$, centered about a curve in the cylindrical surface given by s = const., $-1 \leqslant s \leqslant 1$. We generalize (4.8) by introducing a weighting factor $(1 - s^k)s^n$. We shall illustrate the calculation with $k = 2$, although any even integer such that $k + n \leqslant K$ would suffice. The result is denoted by $\hat{\Gamma}^n_{,t}$ and is given by

$$\begin{aligned}\hat{\Gamma}^n_{,t} &= \frac{1}{2}\int_{-1}^{1}(1 - s^2)s^n\gamma_{,t}(s)\,\mathrm{d}s \\ &= -\frac{1}{2}\int_0^{\ell}\mathrm{d}\lambda\int_{-1}^{1}(1 - s^2)s^n\,\{\mathscr{D}\mathbf{v}^*\cdot\mathbf{x}_{,\lambda}\}_{,s}\,\mathrm{d}s \\ &\quad + \left\{(\eta/4)\int_{-1}^{1}(1 - s^2)s^n\mathscr{D}v^{*3}\,\mathrm{d}s\right\}\Bigg|_{\lambda=0}^{\ell}, \quad n = 0, 1, \ldots, K - 2.\end{aligned} \tag{4.9}$$

Integrating the first term on the right-hand side by parts, we obtain

$$\begin{aligned}\hat{\Gamma}^n_{,t} &= -\frac{1}{2}\int_0^{\ell}\mathrm{d}\lambda\,\{(1 - s^2)s^n\mathscr{D}\mathbf{v}^*\cdot\mathbf{x}_{,\lambda}\}\big|_{s=-1}^{1} \\ &\quad + \frac{1}{2}\int_0^{\ell}\mathrm{d}\lambda\int_{-1}^{1}[ns^{n-1} + (n - 2)s^{n+1}]\{\mathscr{D}\mathbf{v}^*\cdot\mathbf{x}_{,\lambda}\}\,\mathrm{d}s \\ &\quad + \left\{(\eta/4)\int_{-1}^{1}(1 - s^2)s^n\mathscr{D}v^{*3}\,\mathrm{d}s\right\}\Bigg|_{\lambda=0}^{\ell}.\end{aligned} \tag{4.10}$$

The first term in (4.10) is identically zero zince $(1 - s^2) \equiv 0$ at the indicated limits. The second term in (4.10) is composed of two terms which are identical in form to (4.4) and can be expressed by means of (4.7). The last term in (4.10) can be evaluated directly from (2.3). Assembling these results we obtain

$$\begin{aligned}\hat{\Gamma}^n_{,t} &= [n\{-\theta_{n-1}g\zeta - \theta_n g\eta/2 - p_{n-1}/(\varrho\eta)\} \\ &\quad - (n + 2)\{-\theta_{n+1}g\zeta - \theta_{n+2}g\eta/2 - p_{n+1}/(\varrho\eta)\} \\ &\quad - 1/(2\varrho)\{[\hat{p} - (-1)^n\bar{p} - 2np_{n-1}/\eta + \theta_n\varrho g\eta] \\ &\quad - [\hat{p} - (-1)^{n+2}\bar{p} - 2(n + 2)p_{n+1}/\eta + \theta_{n+2}\varrho g\eta]\}]\big|_{\lambda=0}^{\ell} \\ &\equiv 0, \quad n = 0, 1, \ldots, K - 2,\end{aligned} \tag{4.11}$$

since the terms involving p_{n-1}, p_{n+1}, $\hat{p}$, θ_n and θ_{n+2} cancel, and

$$n\theta_{n-1} = (n + 2)\theta_{n+1} = \begin{cases} 1, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}, \end{cases}$$

for level 2 equations or higher.

Equation (4.11) shows that $K - 1$ weighted moments of the time rate of change of cross-sheet circulation for differential contours on a vertical strip are zero (for level 2 equations and higher). Cross-sheet circulation appears not to be preserved in general for level 1 (restricted theory). This result shows that the cross-sheet circulation is preserved best in the center region of the sheet. Higher levels of the theory are required to preserve circulation close to the top and bottom surfaces.

5. Discussion

The results presented in Sections 3 and 4 demonstrate three features of the Kth level theory of thin fluid sheets: that mechanical energy is strictly conserved, that K moments in s of the in-sheet circulation are conserved, and that $K - 1$ weighted moments in s of the cross-sheet circulation are conserved. Of the three results, the cross-sheet circulation appears least well conserved. This is not surprising since the approximation to the flow is made in this direction. At any rate, these three results demonstrate that as K becomes large, we can expect the predicted flow to behave more and more like an ideal fluid. The selection of the level in the hierarchy required for a given problem will depend, of course, on the problem itself and the accuracy required.

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